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Random Clarkson inequalities
and
L_p version of Grothendieck's inequality

by
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Summary. In a recent paper Kato [3] used the Littlewood matrices to generalise Clarkson's inequalities. Our first aim is to indicate how Kato's result can be deduced from a neglected version of the Hausdorff-Young inequality which was proved by Wells and Williams [11]. We next establish "random Clarkson inequalities".. These show that the expected behaviour of matrices whose coefficients are random ± 1's is, as one might expect, the same as the behaviour that Kato observed in the Littlewood matrices. Finally we show how sharp L_p versions of Grothendieck's inequality can be obtained by combining a Kato-like result with a theorem of Bennett [1] on Schur multipliers.

Notational conventions. For convenience we confine our attention to complex Banach spaces. All but the group theoretical results have true analogues in the real case. These analogues are simple corollaries of our theorems . We intend to use the standard notations of Banach space theory. These can be found in the books of Lindenstrauss and Tzafriri [5], Our basic reference for topological group theory is Rudin's book. [9].

1. Kato's inequalities and the Hausdorff-Young theorem.

We begin by interpreting the Littlewood matrices as character tables of appropriate finite abelian groups.

If G is a finite abelian group of order n , the dual group Γ also has order n . The action of $\Gamma = \{\gamma_1, \ldots \gamma_n\}$ on $G = \{g_1, \ldots, g_n\}$ is completely described by an $n\times n$ "character matrix" $C = (c_{ij})$ where

$$c_{ij} = \gamma_i(g_j) \qquad (1 \le i, j \le n).$$

This matrix Does of course depend on the order in which the elements of G and Γ are written down.

Now let $G' = \{g'_i, \ldots g'_m\}$ be another finite abelian group with dual group $\Gamma' = \{\gamma'_i, \ldots \gamma'_m\}$ and character matrix C' . The direct product $G \times G'$ has dual group $\Gamma \times \Gamma'$, and the duality is given by

$$(\gamma,\gamma')((g,g')) - \gamma(g)\gamma'(g') .$$

If the elements of $G \times G'$ are written down in the order

$$(g_1, g'_1), (g, g'_2), \ldots\ldots (g_1, g'_m), (g_2, g'_1), \ldots (g_2, g'_m), \ldots\ldots (g_n, g'_1), \ldots (g_n, g'_m)$$

and if the elements of $\Gamma\times\Gamma'$ are written down in the order

$$(\gamma_1,\gamma'_1), (\gamma_1,\gamma'_2), \ldots (\gamma_1,\gamma'_m), (\gamma_2,\gamma'_1), \ldots, (\gamma_2,\gamma'_m), \ldots (\gamma_n,\gamma'_1), \ldots\ldots (\gamma_n,\gamma'_m)$$

then the character matrix describing the action of $\Gamma\times\Gamma'$ on $G\times G'$ is clearly given by the Kronecker product

$$C \otimes C' = \begin{pmatrix} \gamma_1(g_1)C' & \gamma_1(g_2)C' & \cdots & \gamma_1(g_n)C' \\ \gamma_2(g_1)C' & \gamma_2(g_2)C' & \cdots & \gamma_2(g_n)C' \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \gamma_n(g_1)C' & \gamma_n(g_2)C' & \cdots & \gamma_n(g_n)C' \end{pmatrix}$$

Denote the cyclic group of order 2 by $Z_2 = \{0,1\}$. The dual Groupb is also $Z_2 = \{0,1\}$ and the corresponding charcter matrix is

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now write Z_2^n for the n-fold external direct product of the group Z_2 with itself. We can apply the procedure described above to generate inductively $2^n \times 2^n$ character matrices A_n - $(a_{ij}^{(n)})$ for the groups Z_2^n. Thus

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \; A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix} (n = 1,2,...).$$

These matrices were introduced by Littlewood [6] and are widely known as the Littlewood matrices.

In [11, theorem 1] Wells and Williams proved a generalised Hausdorff-Young theorem- We interpret this theorem in the special case that is relevant to our application,

<u>THEOREM 1</u>. (<u>Wells and Williams</u>) Let G - $\{g_1, \ldots . g_n\}$ be a finite abelian group with dual group $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. Let $x_1 ., \ldots, x_n$ be elements of an L_p (μ) space $(1 \leq p \leq \infty)$. Then

$$\left(\sum_{i=1}^{n} \left\| \sum_{j=1}^{n} \gamma_i (g_j) x_j \right\|_p^{q'} /n \right)^{1/q'} \leq \left(\sum_{j=1}^{n} \| x_j \|_p^q \right)^{1/q}$$

provided that $1 \leq q \leq \min(p,p')$.
[Here, as usual, ' denotes the conjugate exponent.]

<u>Proof</u>. If we equip G with the discrete topology, then the Haar measure on G is simply the counting measure. The Haar measure on the dual Γ will be $1/n$ times the counting measure. Now apply [11, theorem 1].

Kato's inequality is an easy corollary of Theorem 1.

<u>THEOREM 2</u>. (<u>Kato</u>) Let $1 \leq p,r,s \leq \infty$ and let $A = (a_{ij}^{(n)})$ be the n' th Littlewood matrix. Let $x_1, \ldots, x_{2^n}$ be elements of an L_p (μ) space, Then

$$\left(\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} a_{ij}^{(n)} x_j \right\|_p^s \right)^{1/s} \leq 2^{nc(r,s,p)} \left(\sum_{j=1}^{2^n} \| x_j \|_p^r \right)^{1/r}$$

(with the usual convention if r or s is infinite), where

$$c(r,s,p) = \begin{cases} 1/r' + 1/s - \min(1/p') & \text{if } \min(p, p)' \leq r \leq \infty, 1 \leq s \leq \max(p, p') \\ 1/s \quad - - & \text{if } 1 \leq r \leq \min(p, p'), 1 \leq s \leq r' \\ 1/r & \text{if } s' \leq r \leq \infty, \max(p, p') \leq s \leq \infty \end{cases}$$

<u>Proof</u>, We apply Theorem 1 to the group $G = Z^n{}_2$. Then, provided that we list the elements of G and r in the order prescribed in the preamble, we get

$$\gamma_i(g_j) = a_{ij}^{(n)} \quad (1 \leq i, j \leq n) \quad .$$

Consequently , for $1 \leq q \leq \min(p,p')$ we have

$$\sum_{i=1}^{2n} \left(\| \sum_{j=1}^{2n} a_{ij}^{(n)} x_j \|_p^{q'} \right)^{1/q'} \leq 2^{n/q'} \left(\sum_{j=1}^{2n} \| x_j \|_p^{q} \right)^{1/q}$$

We now consider three separate cases.

<u>Case 1</u> . $\min(p,p') \leq r \leq \infty \quad 1 \leq s \leq \max(p,p')$

In this case we set $q = \min(p,p')$. Then, by Hölder's inequality and our basic inequality above, we obtain

$$\begin{aligned} \sum_{i=1}^{2n} \left(\| \sum_{j=1}^{2n} a_{ij}^{(n)} x_j \|_p^{s} \right)^{1/s} &\leq 2^{n(1/s-1/q')} \sum_{i=1}^{2n} \left(\| \sum_{i=1}^{2n} a_{ij}^{(n)} x_j \|_p^{q'} \right)^{1/q'} \\ &\leq 2^{n/s} \left(\sum_{j=1}^{2n} \left\| x_j \right\|_p^{q} \right)^{1/q} \\ &\leq 2^{n(1/s-1/r+1/q)} \left(\sum_{j=1}^{2n} \left\| x_j \right\|_p^{r} \right)^{1/r} \end{aligned}$$

This proves what we want since

$$c(r,s,p) = 1/r' + 1/s - 1/q' = 1/s - 1/r + 1/q \quad .$$

<u>Case 2</u>, $1 \leq r \leq \min(p,p')$, $1 \leq s \leq r'$

In this case we set $q = r$. Then, by Hölder's inequality and the basic inequality above, we obtain

$$\begin{aligned} \sum_{i=1}^{2n} \left(\| \sum_{j=1}^{2n} a_{ij}^{(n)} x_j \|_p^{s} \right)^{1/s} &\leq 2^{n/s - 1/q')} \sum_{i=1}^{2n} \left(\| \sum_{j=1}^{2n} a_{ij}^{(n)} x_j \|_p^{q'} \right)^{1/q'} \\ &\leq 2^{n/s} \left(\sum_{j=1}^{2n} \left\| x_j \right\|_p^{r} \right)^{1/r} \quad \text{(since } q = r\text{).} \end{aligned}$$

<u>Case 3</u>. $s' \leq r \leq \infty$, $\max(p,p') \leq s \leq \infty$

In this case we set $q = s'$. Then, by Hölder's inequality and the basic inequality above, we obtain

$$\begin{aligned} \sum_{i=1}^{2n} \left(\| \sum_{j=1}^{2n} a_{ij}^{(n)} x_j \|_p^{s} \right)^{1/s} . \leq 2^{n/s} &\leq 2^{n/s} \left(\sum_{j=1}^{2n} \left\| x_j \right\|_p^{s'} \right)^{1/s'} \quad \text{(since } q = s' , q' = s\text{)} \\ &\leq 2^{n(1/s+1/s'-1/r)} \left(\sum_{j=1}^{2n} \left\| x_j \right\|_p^{r} \right)^{1/r} \end{aligned}$$

This gives what we want, since $1/s + 1/s' - 1/r = 1/r' = c(r,s,p)$.

The proof of the theorem is now complete.

It should be noted that Kato also gave examples to show that the estimates in Theorem 2 cannot be improved when the measure space has a sufficient supply of mutually disjoint sets of finite positive measure.

We remark also that in [11, section 4] Wells and Williams applied their Hausdorff-Young theorem to the groups Z_2^n, but with a different end in view.

2. Random Clarkson inequalities.

The approach we used in section 1 brought out the underlying algebraic structure of Kato's inequalities. However, Kato's own proof makes no use at all of the algebraic structure. Indeed, Kato uses nothing about the Littlewood matrices, except his ability to calculate their norms as operators from ℓ_r^{2n} to ℓ_s^{2n}. His methods should therefore be capable of yielding results of a non-algebraic nature about more general matrices, provided only that we can calculate their norms as operators from ℓ_r^n to ℓ_s^n.

There is a considerable literature about the norms of a certain class of $n\times n$ matrices when they are considered as operators from ℓ_r^n to ℓ_s^n. This class consists of the matrices whose coefficients are random $\pm$ 1's. The reader might like to consult [1] or [7] for background information. In many situations it turns out that the Littlewood matrices are typical examples of random ± 1 matrices. This situation is no exception.

Before stating our next theorem it is convenient to introduce some notation. If $A = (a_{ij})$ is an $n\times n$ matrix, we write $\|A\|_{r,s}$ for the norm of A as an operator from ℓ_r^n to ℓ_s^n.

THEOREM 3. (Random Clarkson inequalities) Let $1 \leq p,r,s \leq \infty$ and let $A = (a_{ij})$ be an nxn matrix whose coefficients are identically distributed independent random variables taking the values ± 1 with equal probability. Let $x_1,\dots,x$ be elements of an $L_p(\mu)$ space. Then, if E denotes mathematical expectation, we have

$$E\left(\sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} a_{ij}\ddot{x}_j \Big\|_p^s\right)^{1/s} \leq k_n^{c(r,s,p)}\left(\sum_{j=1}^{n} \|x_j\|_p^r\right)^{1/r}$$

where $c(r,s,p)$ is defined as in Theorem 2 and K is a positive absolute constant.

Proof. We first claim that

$$\left(\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n} a_{ij}x_j\Big\|_p^s\right)^{1/s} \le b(r,s,p)\left(\sum_{j=1}^{n}\|x_j\|_p^s\right)^{1/r}$$

$$\text{where } b(r,s,p)=\begin{cases}\|A\|_{r,p}\, n^{1/s-1/p} & \text{if } 1\le r,s\le p\le\infty\\ \|A\|_{r,s} & \text{if } 1\le r\le p\le s\le\infty\\ \|A\|_{p,s}\, n^{1/p-1/r} & \text{if } 1\le p\le r,s\le\infty\\ \|A\|_{p,p}\, n^{1/s-1/r} & \text{if } 1\le s\le p\le r\le\infty\end{cases}$$

The claim can be established by following Kato's proof of his generalised Clarkson inequalities [3, theorem 1]. One merely has to substitute norms of A for norms of Littlewood matrices.

Next, we must deal with the norms of A. This can be done by referring to [1,corollary 3.3] or [7, theorem 1 . 1], where it is shown that

$$E\|a\|_{r,s}\le K\begin{cases} n^{1/r'+1/s-1/2} & \text{if } 2\le r\le\infty, 1\le s\le 2\\ n^{1/s} & \text{if } 1\le r\le 2, 1\le s\le r'.\\ n^{1/r}, & \text{if } s'\le r\le\infty, 2\le s\le\infty\end{cases}$$

Here K is an absolute constant.

The inequality we want now follows if we combine the ingredients above.

Our next result asserts strongly that the random Clarkson inequalities cannot be improved when the L (u) space is big enough.

<u>THEOREM 4</u>. Let $1\le p,r,s\le\infty$ and let $A=(a_{ij})$ be an $n\times n$ matrix whose coefficients are ± 1' s . Let (Ω,Σ,μ) be a measure space with n disjoint sets of finite non-zero measure. Then there exist elements $x_1,\ldots\ldots,x_n$ of $L_p(\mu)$ for which

$$\left(\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n} a_{ij}x_j\Big\|_p^s\right)^{1/s} \ge n^{c(r,s,p)}\left(\sum_{j=1}^{n}\|x_j\|_p^r\right)^{1/r}$$

where c(r,s,p) is defined as in Theorem 2.

<u>Proof</u>. Define a map $T_A : \ell_r^n(L_p) \longrightarrow \ell_r^n(L_p)$ by

$$T_A(x_j)_{1\le j\le n} \longrightarrow \left(\sum_{j=1}^{n} a_{ij}x_j\right)_{1\le i\le n.}$$

We are required to show that the norm of this map is at least $n^{c(r,s,p)}$ Since the transpose of A is also a matrix of ± 1' s and since the norm of T_A is the same as the norm of its transpose, we may confine our attention to the case where $2 \le p \le \infty$. We consider two subsidiary cases.

<u>Case 1</u>. $p' \le r \le \infty$, $1 \le s \le p$

Let $E_1,\ldots\ldots,E_n$ be mutually disjoint sets of finite non-zero measure, Write χ_{E_j} for the characteristic function of E_j $(1 \le j \le n)$. Set

$$x_j = \left|\mu(E_j)\right|^{-1/p} \chi(E_j).$$

$$\text{then} \quad \left(\sum_{i=1}^{n}\Big\| \sum_{j=1}^{n} a_{ij}x_j \Big\|_p^s\right)^{1/s} = \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\|x_j\|_p^p\right)^{s/p}\right)^{1/s} = n^{1/s+1/p} .$$

$$\text{However} \quad \left(\sum_{j=1}^{n}\|x_j\|_p^r\right)^{1/r} = n^{1/r}.$$

$$\text{Consequently} \quad \left(\sum_{i=1}^{n}\Big\| \sum_{j=1}^{n} a_{ij}x_j \Big\|_p^s\right)^{1/s} = n^{1/r'+1/s-1/p'}\left(\sum_{j=1}^{n}\| x_j \|_p^r\right)^{1/r} .$$

This is what we want.

<u>Case 2</u>. $1 \le r \le p'$, $1 \le s \le r'$ or $s' \le r \le \infty$, $p \le s \le \infty$

Choose $z_1,\ldots\ldots,z_n$ with $\sum_{j=1}^{n} |z_j|^r = 1$ such that

$$\|a\|_{r,s} = \left(\sum_{i=1}^{n}\Big|\sum_{j=1}^{n} a_{ij}z_j\Big|^5\right)^{1/s} .$$

Now fix an arbitrary f in $L_p(\mu)$ of norm 1 and set $x_j = z_j f$ $(1 \le j \le n)$.

$$\text{Then} \quad \left(\sum_{i=1}^{n}\Big\| \sum_{j=1}^{n} a_{ij}x_j \Big\|_p^s\right)^{1/s} = \left(\sum_{i=1}^{n}\Big|\sum_{j=1}^{n} a_{ij}z_j\Big|^s\right)^{1/s}$$

$$= \|A\|_{r,s}$$

$$\ge \max(n^{1/s},n^{1/r'}) \qquad \text{by[1,prop.3.2]}$$

$$= n^{c(r,s,p)}\left(\sum_{j=1}^{n}\| x_j \|_p^r\right)^{1/r} .$$

Note that the restriction on the measure space was only needed in case 1.

3. <u>L_p versions of the Grothendieck inequality</u>.

The aim of this section is to find Banach space versions of Grothendieck's famous Hilbert space inequality. For background information on Grothendieck's inequality the reader may consult [5], It is of interest to note that Blei [2.] has considered L_p versions of Grothendieck's inequality. However, his results are rather different.

We begin by giving a slightly unorthodox interpretation of Grothendieck's inequality. It asserts that if $A = (a_{ij})$ is an nxn matrix and if $x_1 \ldots, x_n, y_1, \ldots, y_n$ are elements of the unit ball of a Hilbert space, then

$$\left| \sum_{i,j=1}^{n} a_{ij} < x_j, y_i > \right| \leq 2\|A\|_{\infty,1}.$$

In fact the constant 2 can be improved [8] but for our purposes this is of no great interest. The important thing is that the constant remains bounded as n tends to infinity.

By taking the supremum of the left hand side of Grothendieck's inequality over all valid choices of $y_1, \ldots\ldots, y_n$ we can make the following reinterpretation.

If $A = (a_{ij})$ is an nxn matrix and if $x_1, \ldots, x_n$ are elements of the unit ball of a Hilbert space, then

$$\sum_{i=1}^{n} \left\| \sum_{j=1}^{n} a_{ij} x_j \right\| \leq 2\|A\|_{\infty,1}.$$

One might hope to obtain more general Banach space versions of Grothendieck's inequality. Indeed, if $A = (a_{ij})$ is an $n \times n$ matrix and if $x_1, \ldots, x_n$ are elements of the unit ball of a Banach space E, then the very fact that we are taking finite sums assures us of the existence of a finite positive constant $K_n(E)$ such that

$$\sum_{i=1}^{n} \left\| \sum_{j=1}^{n} a_{ij} x_j \right\| \leq K_n(E)\|A\|_{\infty,1}.$$

(It is understood that $K_n(E)$ is chosen to be the smallest constant for which the inequality holds for all valid choices of A and $x_1, \ldots, x_n$.)

Unfortunately, Lindenstrauss and Pelczynski [4, prop. 5.2] showed Essentially that if E is an infinite dimensional non-Hilbert space, then $K_n(E)$ tends to infinity with n. Thus, in a certain sense, Grothendieck's inequality characterises Hilbert space amongst the infinite dimensional Banach spaces.

However the story is not yet over. We can still ask ourselves how fast $K_n(E)$ tends to infinity. It is evident that $K_n(E) \leq n^2$ for every Banach space E . Our first result refines this crude estimate,

<u>THEOREM 5</u>. $K_n(E) \leq (2n)^{1/2}$ for every Banach space E .

<u>Proof.</u> If $x_1, \ldots, x_n$ are in the unit ball of the Banach space E and if $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} a_{ij} x_j \Big\| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$$

$$\leq n^{\frac{1}{2}} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

$$\leq (2n)^{\frac{1}{2}} \|A\|_{\infty,1}.$$

The last step is Littlewood's inequality with Szarek's constant [10].

We show later that in general Theorem 5 cannot be improved. However, we already know, thanks to Grothendieck, that for suitable Banach spaces Theorem 5 can be improved substantially. Our final theorem gives the growth of K_n E) when E is an $L_p(\mu)$ space. When $E = \ell_p^n$ our result is a trivial consequence of Grothendieck's inequality. However, for other examples of $L_p(\mu)$ spaces the theorem appears to lie deeper.

One constituent of our proof is a theorem of Bennett [1] on Schur multipliers. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices. The Schur product of A with B is another nxn matrix $A*B = (a_{ij} b_{ij})$. If $1 \leq r,s \leq \infty$ we may regard both B and $A*B$ as operators from ℓ_r^n to ℓ_s^n . We define the norm of A as a Schur multiplier from ℓ_r^n to ℓ_s^n by

$$\|A\|_{(r,s)} . = \sup\{ \|A*B\|_{r,s} \;;\; \|B\|_{r,s} \leq 1 \} .$$

Bennett [1, theorem 6.1] that there are positive absolute constants k and K such that for any nxn matrix A we have

$$K \|A\|_{(\infty)} \leq \|A\|_{(2,2)} \leq k \|A\|_{(\infty,1)} .$$

<u>THEOREM 6</u>. If $1 \leq p \leq \infty$ then $K_n(L_p(\mu) \leq M n^{|1/2 - 1/p|}$ where M is a positive absolute constant. Moreover, if the measure space contains n mutually disjoint sets of finite positive measure, then we also have $K_n(L_p(\mu)) \geq \frac{1}{2} n^{|1/2-1/p|}$.

Proof, If A is an $n\times n$ matrix with transpose A^t then $\| A \|_{\infty} = \| A^t \|_{\infty,1}$, so a duality argument assures us that $K_n (L_p(\mu)) = K_n (L_{p'}(\mu))$ • We may therefore restrict attention to the case where $1 \le p \le 2$.

Our first step is to establish an inequality along the lines of those of Kato.

Let $B = (b_{ij})$ be an nxn matrix and let $g_1,\dots,g_n$ be elements of $L_p(\mu)$. Then

$$\left(\sum_{i=1}^{n} \| \sum_{j=1}^{n} b_{ij} g_j \|^2 \right)^{1/2}$$

$$\le \left\| \left(\sum_{i=1}^{n} | \sum_{j=1}^{n} b_{ij} g_j |^2 \right)^{1/2} \right\|_p \quad \text{by Minkowski's inequality since } p \le 2$$

$$\le \|B\|_{2,2} \left\| \left(\sum_{j=1}^{n} | g_j |^2 \right)^{1/2} \right\|_p \quad \text{by the definition of } \|B\|_{2,2}$$

$$\le \|B\|_{2,2} \left\| \left(\sum_{j=1}^{n} | g_j |^p \right)^{1/p} \right\|_p \quad \text{since } p \le 2$$

$$\le \|B\|_{2,2} \left(\sum_{j=1}^{n} \| g_j \|_p^p \right)^{1/p}$$

$$\le n^{1/p-1/2} \|B\|_{2,2} \left(\sum_{j=1}^{n} \|g_j\|_p^2 \right)^{1/2} \quad \text{by Hölder's inequality.}$$

Thus if we write

$$S=\sup\left\{ \left(\sum_{i=1}^{n} \| \sum_{j=1}^{n} b_{ij} g_j \|_p^2 \right)^{1/2} : \left(\sum_{j=1}^{n} \|g_j\|_p^2 \right)^{1/2} \le 1,\ \|B\|_{2,2} \le 1 \right\}$$

we can assert that

$$s \le n^{1/p-1/2}.$$

Next we look for a lower bound for S . Simple manipulations give

$$S = \sup\left\{ | \sum_{i,j=1}^{n} b_{ij} < f_i, g_j > s_i t_j | \right\}$$

where the supremum is taken over all $\|B\|_{2,2} \le 1$, all $\| f_i \|_{p'} \le 1$, all $\|g_j\|_p \le 1$, all $\sum_{i=1}^{n} | s_i | \le 1$ and all $\sum_{j=1}^{n} | t_j |^2 \le 1$.

But now this has a simple interpretation in terms of Schur multipliers which will enable us to use Bennett's theorem.

$S = \sup\{\| <f_i, g_j> \|_{(2,2)} : \| f_i \|_{p'} \le 1, \ \| g_j \|_p \le 1 \ (1 < i, j \le n)\}$
$\ge k. \sup\{ \| < f_i, g_j > \|_{(\infty,1)} \ \| f_i \|_{p'} \le \| g_j \|_p \le 1 \ (1 \le i, j \le n)\}$

$$= k. \sup\left\{\left|\sum_{i,j=1}^{n} a_{ij} < y_i, x_j >\right|\right\}$$

where the final supremum is taken over all nxn matrices $A = (a_{ij})$ with $\| A \|_{\infty\text{-}1} \le 1$, over all $x_1, \ldots, x_n$ in the unit ball of $L_p(\mu)$ and over all $y_1, \ldots, y_n$ in the unit ball of $L_{p'}(\mu)$.

Thus we have shown that

$$s \ge k.\sup\left\{\sum_{i=1}^{n} \| \sum_{j=1}^{n} a_{ij} x_j \|_p : \|A\|_{\infty,1} \le 1, \ \left\|x_j\right\|_p \le 1 (1 \le i \le n)\right\}$$

where k is an absolute constant. If we now combine the upper and lower estimates for S we obtain

$$\sum_{i=1}^{n} \| \sum_{j=1}^{n} a_{ij} x_j \|_p \le k^{-1} n^{1/p-1/2} \|A\|_{\infty,1}$$

for all $n \times n$ matrices $A = (a_{ij})$ and for all $x_1, \ldots, x_n$ in the unit ball of $L_p(\mu)$. This was our first objective.

The second objective follows from Kato's results. He showed that if $n = 2^m$ and if the measure space contains n mutually disjoint sets of finite positive measure, then $K_n(L_p(\mu)) \ge n^{|1/p-1/2|}$. He does this by proving that if $A - (a_{ij}^{(m)})$ is the 2 x2 Littlewood matrix, then there is a choice of $x_1, \ldots, x_{2m}$ in the unit ball of $L_p(\mu)$ for which

$$\sum_{i=1}^{2m} \| \sum_{j=1}^{2m} a_{ij}^{(m)} x_j \|_p = 2^m \ |^{1/p-1/2|} \ \|A_m\|_{\infty,1}.$$

If n is not a power of 2, let 2^m be the largest power of 2 less than n. Augment the Littlewood matrix A_m with zeros to obtain an nxn matrix $A = (a_{ij})$. Then $\| A_m \|_{\infty-1} \ \|A\|_{\infty,1}$. Choose $x_1, \ldots, x_{2m}$ as above and take arbitrary $x_{2m+1}, \ldots, x_n$ in the unit ball of $L_p(\mu)$. Then

$$\sum_{i=1}^{n} \| \sum_{j=1}^{n} a_{ij} x_j \|_p = \sum_{i=1}^{2m} \| \sum_{j=1}^{2m} a_{ij}^{(m)} x_j \|_p = 2^{m|1/p-1/2|} \|A_m\|_{\infty,1}$$

$$= 2^{m|1/p/2|} \|A\|_{\infty,1} \ge \frac{1}{2} n^{|1/p-1/2|} \|A\|_{\infty,1}$$

This completes the proof.

Theorem 6, with $p = 1$ or $p = \infty$, shows that Theorem 5 cannot be improved. In fact, it is easy to see that $K_n(\ell_1) = (2n)^{1/2}$.

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